AD-A096 648 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
SOME EXISTENCE THEOREMS FOR SEMILINEAR HYPERBOLIC SYSTEMS IN ON--ETC(U)
JAN 81 L C TARTAR DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2164 NL

1 OF 1
AD A
096648

END
DATE
FILMED
4-81
DTIC

LEVEL II (2)

AD A096648

MRC Technical Summary Report #2164

SOME EXISTENCE THEOREMS FOR SEMILINEAR HYPERBOLIC SYSTEMS IN ONE SPACE VARIABLE

Luc C. Tartar

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

January 1981

(Received September 2, 1980)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

SOME EXISTENCE THEOREMS FOR SEMILINEAR HYPERBOLIC SYSTEMS IN ONE SPACE VARIABLE

Luc C. Tartar

Technical Summary Report #2164
January 1981

ABSTRACT

We consider semilinear hyperbolic systems in one space variable of the type

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + C_i \dfrac{\partial u_i}{\partial x} + \sum_{j,k} A_{ijk} u_j u_k = 0, & x \in R, \quad t \in R, \quad i = 1,\dots,p \\ u_i(x,0) = \varphi_i(x), & x \in R, \quad i = 1,\dots,p \,. \end{cases}$$

We first introduce a special condition

(S) $$A_{ijk} = 0 \quad \text{if} \quad C_j = C_k \,.$$

Under condition (S) we prove: local existence and uniqueness if the data are in $L^1(\mathbb{R})$; global existence, L^∞ stability and the existence of wave operators and of a scattering operator when the data have small norm in $L^1(\mathbb{R})$.

Adding a sign condition

(s) $$A_{ijk} \leq 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad i \neq k$$

and the entropy condition

(E) $\sum_{i,j,k} A_{ijk}\lambda_j\lambda_k \operatorname{Log} \lambda_i \geq 0$ for all $\lambda \in \mathbb{R}^p$ such that $\lambda_i > 0$ for each i we obtain global existence if the data are nonnegative and in $L^\infty(\mathbb{R})$.

We then replace condition (S) by a weaker one and obtain some of the above results in that case.

AMS (MOS) Subject Classification: 35L99

Key Words: Semilinear hyperbolic system, Kinetic theory, Global existence, Asymptotic behaviour, Scattering

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

We study semilinear hyperbolic systems with quadratic nonlinearities which originate in the kinetic theory of gas as a simplification of Boltzmann's equation.

Local existence is well known for these equations and the main problem is to prove global existence for nonnegative bounded data.

Except for the unrealistic case where a bounded invariant region exists, no result of this type is known in three space dimensions. As in all preceding results, based on the work of Mimura-Nishida and Crandall-Tartar, we restrict ourselves to one space dimension. We show global existence for a quite general class of systems and under some special condition (S) we obtain information on the asymptotic behaviour and on scattering when the data have small L^1 norm.

The new idea lies in the introduction of some functional spaces where some products can be defined; this enables us to define an appropriate notion of solution in L^1 and then use it to obtain local and global existence for data in $L^1(R)$.

SOME EXISTENCE THEOREMS FOR SEMILINEAR HYPERBOLIC SYSTEMS IN ONE SPACE VARIABLE

Luc C. Tartar

0. Introduction

The motivation for considering the systems studied here comes from kinetic theory of gases where, by allowing only a discrete set of velocities in the Boltzmann equation, one obtains a semilinear system where the quadratic nonlinearity corresponds to interaction between particles through collisions.

Although the nonlinearities considered here correspond to realistic models we are not able to solve the real three-dimensional problem and we restrict the analysis to one dimensional situations (except for the unrealistic cases where there is a bounded invariant set one is still waiting for a global existence theorem in L^∞ in more than one space dimension).

Following an argument due to M. G. Crandall and the author the global existence in L^∞ follows (by using finite propagation speed, nonnegativeness of solutions and entropy) from an estimate, similar to one obtained by Mimura-Nishida, in which one proves a global L^∞ bound for data in L^∞ with small L^1 norm.

To prove the desired estimates we introduce some functional space modeled on L^1 and the crucial remark is that one can still define some products and give a meaning to solutions corresponding to L^1 initial data.

This analysis is first carried on for special quadratic terms and then to a more general case. Apart from simplifying the analysis, the special case arise naturally as the only one (excluding the linear case) for which the semi-group defining the solution is (sequentially) continuous for the weak-star topology in L^∞.

Some results concerning asymptotic behaviour and scattering are also proved.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

1. Statement of the problem

We will consider the following system

$$(1.1)\quad \begin{cases} \dfrac{\partial u_i}{\partial t} + C_i \dfrac{\partial u_i}{\partial x} + \sum\limits_{j,k} A_{ijk}u_j u_k = 0, \quad x \in R, \quad t \in I, \quad i = 1,\dots,p \\ u_i(x,0) = \varphi_i(x), \quad x \in R, \quad i = 1,\dots,p\ , \end{cases}$$

where the C_i are real constants (not necessarily distinct), the A_{ijk} are real constants (satisfying $A_{ijk} = A_{ikj}$ for every i, j, k) and the time interval I contains 0.

The original problem from kinetic theory of gases is

$$(1.2)\quad \begin{cases} \dfrac{\partial u_i}{\partial t} + \text{grad } u_i \cdot v_i + \sum\limits_{j,k} A_{ijk}u_j u_k = 0, \quad x \in R^3, \quad t \in I, \quad i = 1,\dots,p \\ u_i(x,0) = \varphi_i(x), \quad x \in R^3, \quad i = 1,\dots,p\ , \end{cases}$$

where $u_i(x,t)$ is the density of particles having velocity v_i (the v_i are here distinct vectors in R^3) and the quadratic term corresponds to interaction through collisions (some information given by physics on the coefficients A_{ijk} is useful for the mathematical treatment of the equation).

In the case where the initial data only depends upon $x \cdot n$ (where n is some unit vector) the solution will be a function of $x \cdot n$ and t and will satisfy (1.1) with $C_i = v_i \cdot n$.

We now state briefly some classical results concerning (1.1) and (1.2):

a) If the data φ_i are in L^∞ then there is a local solution in time.

b) The system exhibits a finite propagation speed: $u(x,t)$ only depends upon $\varphi(x - tw)$ with $w \in [\min C_i, \max C_i]$ for (1.1) and $w \in$ convex hull$(v_1,\dots,v_p)$ for (1.2).

c) The solution $u_i(x,t)$ is nonnegative for all x, i and $t > 0$ if each φ_i is nonnegative and if the coefficients A_{ijk} satisfy

$$(1.3) \qquad A_{ijk} \leq 0 \text{ if } i \neq j \text{ and } i \neq k$$

(particles of velocity v_i can only be created in collisions where they do not enter).

d) Conservation of mass holds if the coefficients A_{ijk} satisfy

$$(1.4) \qquad \sum_i A_{ijk} = 0 \quad \text{for every } j,k \ .$$

Then if the data φ_i are in $L^\infty \cap L^1$ the solution satisfies

$$(1.5) \qquad \frac{d}{dt} \sum_i \int u_i(x,t)dx = 0 \ .$$

If (1.3) and (1.4) hold, the L^1 bound on the solution only depends on the L^1 bound of the data (we still need $\varphi_i \geq 0$ and $\varphi_i \in L^\infty \cap L^1$) as long as the solution stays in L^∞.

e) The entropy condition is related to the following condition

$$(1.6) \qquad \sum_{i,j,k} A_{ijk}\lambda_j\lambda_k(\text{Log } \lambda_i + 1) \geq 0 \quad \text{for all } \lambda \text{ such that } \lambda_i > 0 \text{ for all } i \ .$$

Then if (1.3) and (1.6) hold and if the data φ_i are nonnegative L^∞ with compact support the solution satisfies

$$(1.7) \qquad \frac{d}{dt} \sum_i \int u_i(x,t)\text{Log } u_i(x,t)dx \leq 0 \quad \text{for } t > 0 \ .$$

From these results one sees that L^1 is a natural space to use; unfortunately as the quadratic terms are not defined on this space, we have to impose a restriction on solutions and we are led to the following definition.

<u>Definition 1</u>: If $\varphi_i \in L^1(\mathbb{R})$ for each i, a solution of (1.1) is an element $u = (u_1,\ldots,u_p)$ such that

α) u_i is continuous on $\bar{I}$ with values in $L^1(\mathbb{R})$ for each i,

β) $\frac{\partial u_i}{\partial t} + c_i \frac{\partial u_i}{\partial x} \in L^1(\mathbb{R} \times I)$ for each i,

γ) $u_j u_k \in L^1(\mathbb{R} \times I)$ for each j, k such that $A_{ijk} \neq 0$ for some i,

δ) u satisfies (1.1) (each term having a meaning using α, β, γ). ∎

<u>Remark 1</u>: In β) derivatives are taken in the sense of distributions; an equivalent statement is

β') Let $w_i(x,t) = u_i(x - C_i t, t)$. w_i is absolutely continuous on I with values in $L^1(\mathbb{R})$ for each i.

2. Statement of the results under special condition (S)

We will make an extensive use of the following special condition:

(S) $$A_{ijk} = 0 \quad \text{if} \quad C_j = C_k \;.$$

The relation of condition (S) with respect to continuity in the L^∞ weak-star topology is considered in the appendix.

Theorem 1: (Global existence for small L^1 data). Assume (S) holds. Then there exists $E_0 > 0$, $k_1 > 0$, $k_2 > 1$ such that:

a) If $\varphi_i \in L^1(\mathbb{R})$ for each i and satisfy

$$\sum_i \|\varphi_i\|_{L^1(\mathbb{R})} < E_0 \tag{2.1}$$

then there is a unique solution of (1,1) on interval $I = (-\infty,+\infty)$; this solution satisfies

$$\sum_i \left\|\frac{\partial u_i}{\partial t} + C_i \frac{\partial u_i}{\partial x}\right\|_{L^1(\mathbb{R}\times I)} < k_1 \sum_i \|\varphi_i\|_{L^1(\mathbb{R})} \;. \tag{2.2}$$

b) If $\bar\varphi_i \in L^1(\mathbb{R})$ with $\sum_i \|\bar\varphi_i\|_{L^1(\mathbb{R})} < E_0$, corresponding to solution $\bar u$ then

$$\sup_{t\in\mathbb{R}} \sum_i \|u_i(\cdot,t) - \bar u_i(\cdot,t)\|_{L^1(\mathbb{R})} < k_2 \sum_i \|\varphi_i - \bar\varphi_i\|_{L^1(\mathbb{R})} \;. \tag{2.3}$$ ■

Theorem 2: (Local existence for L^1 data). Assume (S) holds. Let $\varphi_i \in L^1(\mathbb{R})$ for each i then there exists $t_0 > 0$ such that (1.1) has a (unique) solution on interval $I = (-t_0,t_0)$. ■

Theorem 3: (L^∞ regularity for small L^1 data). Assume (S) holds. Then there exists $E_1 > 0$, $k_3 > 1$ such that: if $\varphi_i \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and satisfy

$$\sum_i \|\varphi_i\|_{L^1(\mathbb{R})} < E_1. \tag{2.4}$$

Then the solution is essentially bounded in $R \times R$ and satisfies

$$\sup_{t,i} \|u_i(\cdot,t)\|_{L^\infty(\mathbb{R})} < k_3 \max_i \|\varphi_i\|_{L^\infty} \;. \tag{2.5}$$ ■

Theorem 4: (Global L^∞ existence). Assume (S), (1.3), and (1.6) hold. Then there exists a growth function $F(t,M)$ such that: if the data satisfy $0 \leq \varphi_i(x) \leq M$ a.e. for each i, then the solution exists for $t \in [0,\infty[$ and satisfies $0 \leq u_i(x,t) \leq F(t,M)$ a.e. for each i. ∎

Theorem 5: *(Asymptotic behaviour for small L^1 data). Assume (S) holds.* Let $\varphi_i \in L^1(\mathbb{R})$ for each i with $\sum_i \|\varphi_i\|_{L^1(\mathbb{R})} \leq E_0$. Let $v_i(x,t) = u_i(x + C_i t, t)$. Then as t goes to $+\infty$ or $-\infty$, v_i has strong limits in $L^1(\mathbb{R})$. If moreover the solution is bounded in $L^\infty(\mathbb{R})$ for $t \geq 0$ then v_i has a strong limit in $L^\infty(\mathbb{R})$ as t goes to $+\infty$ (similarly a bound for $t \leq 0$ gives a strong limit as t goes to $-\infty$). ∎

If we denote by $S_0(t)\varphi$ the solution corresponding to the linear case (all $A_{ijk} = 0$) and $S(t)\varphi$ the solution of our problem (defined for φ with a small L^1 norm), then Theorem 5 says that the wave operators $W\pm = \lim_{t\to\pm\infty} S_0(-t)S(t)$ exist for φ in a suitable domain. Similarly one can look at limits of $S(t)S_0(-t)$.

Theorem 6: Assume (S) holds. Let $\varphi_i \in L^1(\mathbb{R})$ for each i with $\sum_i \|\varphi_i\|_{L^1(\mathbb{R})} \leq E_0$. Let $v_i^T(x,t)$ be the solution of (1.1) with initial data $\psi_i^T(x) = \varphi_i(x - TC_i)$; then as T goes to $\pm\infty$, $v_i^T(x,T)$ has strong limits in $L^1(\mathbb{R})$. ∎

Let (2.6) $D_m = \left\{\varphi : \varphi \in (L^1(\mathbb{R}))^p : \sum_j \|\varphi_j\|_{L^1(\mathbb{R})} \leq \dfrac{E_0}{(1 + k_1)^m}\right\}$.

Theorem 7: (Scattering). Assume (S) holds. The wave operators $W_\pm = \lim_{t\to\pm\infty} S_0(-t)S(t)$ map D_m into D_{m-1} for each $m \geq 0$; $M_\pm = \lim_{t\to\pm\infty} S(t)S_0(-t)$ map D_m into D_{m-1} for each $m \geq 0$. If A is any of the four operators $W_\pm$, $M_\pm$ then

$$(2.7)\qquad \begin{cases} \|A\varphi - A\psi\| \leq k_2\|\varphi - \psi\|, & \varphi,\psi \in D_0 \\ \|A\varphi - A\psi\| \geq \dfrac{1}{k_2}\|\varphi - \psi\|, & \varphi,\psi \in D_1, \end{cases}$$

$$(2.8)\qquad M_+W_+,\ W_+M_+,\ M_-W_-,\ W_-M_- = \text{identity on } D_1 .$$

One can define the wave operator $S = W_+M_-$ mapping D_m into D_{m-2} for $m \geq 1$ satisfying

$$(2.9)\qquad W_+ = SW_- \text{ on } D_2 ,$$

$$
(2.10)\qquad \begin{cases} \|S\varphi - S\psi\| \le k_2^2 \|\varphi - \psi\|, & \varphi,\psi \in D_1 \\[2ex] \|S\varphi - S\psi\| \ge \dfrac{1}{k_2^2} \|\varphi - \psi\|, & \varphi,\psi \in D_2 \,. \end{cases}
$$

S is one to one on D_2 and its inverse $S^{-1} = W_- M_+$ maps D_m into D_{m-2} for $m \ge 1$ satisfies (2.10) and $SS^{-1} = S^{-1}S =$ identity on D_3. ∎

3. Functional spaces for (1.1)

Let I be a time interval containing 0 and J be an interval of $\mathbf{R}$. We consider functions defined on domain

(3.1) $$D = \{(x,t) \in R \times I \text{ such that } x - C_k t \in J \text{ for } k = 1,\dots,p\} .$$

If $f \in L^1(D)$ and $g \in L^1(J)$ one can then solve the problem

(3.2) $$\begin{cases} \dfrac{\partial v}{\partial t} + C_j \dfrac{\partial v}{\partial x} = f, & (x,t) \in D \\ v(x,0) = g, & x \in J \end{cases}$$

whose solution is given by

(3.3) $$v(x,t) = g(x - C_j t) + \int_0^t f(x - C_j s, t - s) ds \quad \text{a.e.} \quad (x,t) \in D .$$

Let us define the following space

(3.4) $$\begin{cases} V_j = \{v \text{ defined on } D \text{ satisfying (3.2) with } f \in L^1(D) \text{ and } g \in L^1(J)\} \\ \qquad\qquad \text{with norm } |||v|||_{V_j} = \|f\|_{L^1(D)} + \|g\|_{L^1(J)} . \end{cases}$$

V_j is a Banach space isometric to $L^1(D) \times L^1(J)$ (functions in V_j are in $L^1_{loc}(D)$ and are only defined almost everywhere); all properties can be checked on smooth functions (f and g can be approached by smooth functions with compact support, but the support of the corresponding v will not be compact).

<u>Lemma 1</u>: If $v \in V_j$ then there exists $w \in L^1(J)$ with

(3.5) $$\begin{cases} |v(x,t)| \leq w(x - C_j t) \quad \text{a.e. in } D \\ \|w\|_{L^1(J)} = |||v|||_{V_j} . \end{cases}$$

<u>Proof</u>: By (3.3) $|v(x,t)| \leq |g(x - C_j t)| + \int_0^t |f(x - C_j s, t - s)| ds$. Let $K_y = \{\tau \in I : (y + C_j \tau, \tau) \in D\}$ for $y \in J$ and define

$$w(y) = |g(y)| + \int_{K_y} |f(y + C_j\tau,\tau)|d\tau$$

then we have $|v(x,t)| \leq w(x - C_j t)$ and $\|w\|_{L^1(J)} = |||v|||_{V_j}$. ∎

<u>Lemma 2</u>: If $v_j \in V_j$ and $v_k \in V_k$ with $C_j \neq C_k$ then $v_j v_k \in L^1(D)$ and

(3.6) $$\|v_j v_k\|_{L^1(D)} \leq \frac{1}{|C_j - C_k|} |||v_j|||_{V_j} |||v_k|||_{V_k} .$$

<u>Proof</u>: By (3.5) it is enough to bound $\int_D w_j(x - C_j t)w_k(x - C_k t)dxdt$. By the change of variable $y = x - C_j t$, $z = x - C_k t$ this integral is $\frac{1}{|C_j - C_k|} \int_{D'} w_j(y)w_k(z)dydz$ which is less than $\frac{1}{|C_j - C_k|} \|w_j\|_{L^1(J)} \|w_k\|_{L^1(J)}$. ∎

<u>Remark 2</u>: $C_j \neq C_k$ is necessary in Lemma 2 as it is impossible to bound v^2 in $L^1(D)$ for $v \in V_j$ (take $v = w(x - C_j t)$ with $w \in L^1(J)$).

The map $v \to v(\cdot,t_0)$ is continuous from V_j onto $L^1(D^1\{t = t_0\})$ and so one cannot define the product $v_j(\cdot,t_0)v_k(\cdot,t_0)$ for every t_0. ∎

<u>Notation</u>: $V = \{v = (v_1,\ldots,v_p)$ such that $v_j \in V_j$ for each $j\}$ the norm on V being $\sum_j |||v_j|||_{V_j}$.

4. Proofs of Theorems 1 to 7

We first remark that u is a solution of (1.1) in the sense of Definition 1 if and only if each u_j belongs to V_j (defined with $J = R$) and u satisfies equation (1.1).

Indeed if u is a solution then by β) each u_j is in V_j. Conversely if $u_j \in V_j$ for each j, then by Lemma 2 the products $u_j u_k$ belong to $L^1(D)$ if $C_j \neq C_k$; but, from condition (S) the only products $u_j u_k$ appearing in Equation (1.1) satisfy $C_j \neq C_k$ and so each term is in $L^1(D)$.

4.a: Let D be defined like in (3.1) and let $\varphi_i \in L^1(J)$ for each i; we want to find a solution of (1.1) on D as a consequence of the fixed point theorem for strict contractions.

Construct a mapping from $L^1(J)^p \times V$ into V as follows:

$$(4.1)\quad \begin{cases} \text{For } \varphi_i \in L^1(J) \text{ for each } i \text{ and } u \in V,\ v = A(\varphi,u) \text{ is the solution of} \\ \dfrac{\partial v_i}{\partial t} + C_i \dfrac{\partial v_i}{\partial x} + \sum\limits_{j,k} A_{ijk} u_j u_k = 0, \quad i = 1,\dots,p,\ (x,t) \in D \\ v_i(x,0) = \varphi_i(x), \quad i = 1,\dots,p,\ x \in J\ . \end{cases}$$

By condition (S) and Lemma 2 the sums $\sum\limits_{j,k} A_{ijk} u_j u_k$ are in $L^1(D)$ for each i and so $v \in V$. u is a solution of (1.1) if and only if it is a fixed point of the map $u \to A(\varphi,u)$. We will show that, if $\sum\limits_i \|\varphi_i\|_{L^1(J)}$ is small enough, this map is a strict contraction on some closed set of V.

Let $\varepsilon_j = \|\varphi_j\|_{L^1(J)}$ and $E = \sum\limits_j \varepsilon_j$.

Let $\alpha_j = \left\|\dfrac{\partial u_j}{\partial t} + C_j \dfrac{\partial u_j}{\partial x}\right\|_{L^1(D)}$ and $\beta_j = \left\|\dfrac{\partial v_j}{\partial t} + C_j \dfrac{\partial v_j}{\partial x}\right\|_{L^1(D)}$. Using equation (4.1) and Lemma 2 we have

$$(4.2)\qquad \beta_i \le \sum_{j,k}{}' \frac{|A_{ijk}|}{|C_j - C_k|} (\varepsilon_j + \alpha_j)(\varepsilon_k + \alpha_k)$$

(where Σ' design sums without the undefined terms $\frac{0}{0}$ corresponding to $C_j = C_k$). Define

$$(4.3)\qquad \gamma = \max_{j,k} \sum_i{}' \frac{|A_{ijk}|}{|C_j - C_k|}\ .$$

By summing (4.2) in i one gets $\sum_i \beta_i < \gamma(E + \sum_j \alpha_j)^2$. Consider the following closed set of V:

$$(4.4)\qquad \begin{cases} B_{\varphi,r} = \{v \in V : v_i(x,0) = \varphi_i(x), \quad i = 1,\dots,p,\ x \in J, \\ \sum_i \left\| \frac{\partial v_i}{\partial t} + c_i \frac{\partial v_i}{\partial x} \right\|_{L^1(D)} < r\} \end{cases}$$

the preceding inequality implies that A maps $B_{\varphi,r}$ into $B_{\varphi,s}$ if $s > \gamma(E + r)^2$. Now we bound the Lipschitz constant of A on $B_{\varphi,r}$: Let $\bar{u} \in B_{\varphi,r}$ and $\bar{v} = A(\varphi,\bar{u})$ and $\bar{\alpha}_j = \left\| \frac{\partial \bar{u}_i}{\partial t} + c_i \frac{\partial \bar{u}_i}{\partial x} \right\|_{L^1(D)}$. Then

$$(4.5)\qquad \begin{aligned} \left\| \frac{\partial}{\partial t}(v_i - \bar{v}_i) + c_i \frac{\partial}{\partial x}(v_i - \bar{v}_i) \right\|_{L^1(D)} &< \sum_{j,k}' \frac{|A_{ijk}|}{|c_j - c_k|} [(\varepsilon_j + \alpha_j)|||u_k - \bar{u}_k|||_{V_k} \\ &+ |||u_j - \bar{u}_j|||_{V_j}(\varepsilon_k + \bar{\alpha}_k)] \end{aligned}$$

and so $\sum_i |||v_i - \bar{v}_i|||_{V_i} < 2\gamma(E + r) \sum_j |||u_j - \bar{u}_j|||_{V_j}$. Thus

$$(4.6)\qquad \begin{cases} A \text{ maps } B_{\varphi,r} \text{ into } B_{\varphi,s} \text{ with } s = \gamma(E + r)^2 \text{ with} \\ \text{Lipschitz constant } 2\gamma(E + r)\,. \end{cases}$$

Define E_0 by

$$(4.7)\qquad 4\gamma E_0 < 1\,.$$

So we have shown, by taking $r = E$, that

(4.8) If $\sum_j \varepsilon_j = E < E_0$ then A is a strict contraction on $B_{\varphi,E}$.

So we have proved the existence part in Theorem 1 (taking $J = R$) and inequality (2.2) with $k_1 = 1$.

Let $\varphi_j, \bar{\varphi}_j \in L^1(J)$ for each j with $\sum_j \|\varphi_j\|_{L^1(J)} < E_0$, $\sum_j \|\bar{\varphi}_j\|_{L^1(J)} < E_0$. Let $u \in B_{\varphi,E_0}$, $\bar{u} \in B_{\bar{\varphi},E_0}$, $v = A(\varphi,u)$ and $\bar{v} = A(\bar{\varphi},\bar{u})$. Inequality (4.5) is still true if we

replace $\varepsilon_k + \bar{\alpha}_k$ by $\bar{\varepsilon}_k + \bar{\alpha}_k$ and it gives, after summing in i:

$$|||v - \bar{v}|||_V \leq \sum_i \|\varphi_i - \bar{\varphi}_i\|_{L^1(J)} + 4\gamma E_0 |||u - \bar{u}|||_V .$$

If u and $\bar{u}$ are the solutions corresponding to φ and $\bar{\varphi}$ we obtain

$$|||u - \bar{u}|||_V \leq \frac{4\gamma E_0}{1 - 4\gamma E_0} \sum_i \|\varphi_i - \bar{\varphi}_i\|_{L^1(J)} \tag{4.9}$$

which proves inequality (2.3) with $k_2 = \dfrac{4\gamma E_0}{1 - 4\gamma E_0}$. Uniqueness is so far only proved for solutions lying in B_{φ, E_0}.

4.b: Let u, $\bar{u}$ be two solutions on $R \times I$ corresponding to the same data $\varphi \in L^1(J)^P$. We want to show that u and $\bar{u}$ coincide on $[-t_0, t_0]$ for some $t_0 > 0$ (this will prove uniqueness because the set of t such that $u(\cdot,t) = \bar{u}(\cdot,t)$ will then be opened and closed).

As $\dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x}$ and $\dfrac{\partial \bar{u}_i}{\partial t} + c_i \dfrac{\partial \bar{u}_i}{\partial x}$ are in $L^1(R \times I)$ for each i, one can find $\delta > 0$ such that, if $D \subset R \times I$ has a measure less than δ, then

$$\sum_i \left\|\frac{\partial u_i}{\partial t} + c_i \frac{\partial u_i}{\partial x}\right\|_{L^1(D)} < E_0 \quad \text{and} \quad \sum_i \left\|\frac{\partial \bar{u}_i}{\partial t} + c_i \frac{\partial \bar{u}_i}{\partial x}\right\|_{L^1(D)} < E_0.$$

Similarly there exists δ' such that, if $J \subset R$ has a measure less than δ', then

$$\sum_i \|\varphi_i\|_{L^1(J)} < E_0.$$

Then for $r > 0$ small enough take $J = [x_0, x_0 + r]$, D as in (3.1); then $u, \bar{u} \in B_{\varphi, E_0}$ and thus they coincide on D by step 1. By moving x_0 on R u and $\bar{u}$ coincide in a strip $R \times [-t_0, t_0]$.

So the uniqueness part is proved for Theorems 1 and 2.

4.c: If $\varphi_j \in L^1(\mathbb{R})$ for each j, then one can find a finite number of intervals J_α, $\alpha = 1, \ldots, q$ such that $\bigcup_\alpha J_\alpha^0 = R$ and

$$\sum_j \|\varphi_j\|_{L^1(J_\alpha)} < E .$$

Let D_α be as in (3.1) with J replaced by J_α; let u_α be the corresponding solution in D_α. If $J_\alpha \cap J_\beta \neq \emptyset$ then on $D_\alpha \cap D_\beta$ we have two solutions u_α and u_β which correspond to the same small data and thus they coincide. As $\bigcup_\alpha D_\alpha$ contains a strip $\mathbb{R} \times [-t_0, t_0]$ we can glue the u_α together to obtain a solution in the strip and thus Theorem 2 is proved.

4d: Let $J \subset R$ and D as in (3.1). Let $\varphi_i \in L^1(J) \cap L^\infty(J)$ for each i and define $M(t) = \text{ess sup}\{|u_j(x,s)|;\ j = 1,\dots,p,\ (x,s) \in D,\ |s| \le t\}$ so $M(0) = \max_i \|\varphi_i\|_{L^\infty(J)}$; we want to bound $M(t)$ in terms of $M(0)$. We know that $u_i(x,t) = \varphi_i(x - C_i t) - \sum_{j,k} \int_0^t A_{ijk} u_j u_k (x - C_i s, t - s) ds$. For $C_j \neq C_k$ we want to bound the integral

$$m = \left|\int_0^t |u_j u_k (x - C_i s, t - s)| ds\right| . \tag{4.10}$$

We can assume that $C_j \neq C_i$ (as C_j and C_k cannot be both equal to C_i); we bound $|u_k(x - C_i s, t - s)|$ by $M(t)$ and $\left|\int_0^t |u_j(x - C_i s, t - s)| ds\right|$ by $\frac{1}{|C_i - C_j|} |||u_j|||_{V_j}$ so

$$m \le \frac{M(t)}{|C_i - C_j|} |||u_j|||_{V_j} \tag{4.11}$$

and thus

$$|u_i(x,t)| \le \|\varphi_i\|_{L^\infty(J)} + \kappa M(t) \sum_j |||u_j|||_{V_j}$$

and so $M(t) \le M(0) + \kappa M(t) |||u|||_V$. But we know that if $\sum_j \|\varphi_j\|_{L^1(J)} \le E_1 \le E_0$ we have $|||u|||_V \le 2E_1$ and so if $2\kappa E_1 < 1$ we obtain $M(t) \le \frac{1}{1 - 2\kappa E_1} M(0)$. This estimate is valid as long as the solution is bounded, but the bound obtained being independent of t we have global existence in L^∞ and Theorem 3 is proved.

4.e: To prove global existence in Theorem 4 we will use the entropy and this requires nonnegativeness of the solution.

Let A_{ijk} satisfy the sign condition (1.3). Let $\varphi_i \in L^\infty(\mathbb{R})$ for each i: If $\varphi_i \ge 0$ for each i, then the solution satisfies $u_i(x,t) \ge 0$ for $t \ge 0$, $x \in R$ and $i = 1,\dots,p$. [One can show this by using a different fixed point argument and consider

$v = B(\varphi,u)$ defined by

$$(4.12)\qquad \begin{cases} \dfrac{\partial v_i}{\partial t} + C_i \dfrac{\partial v_i}{\partial x} + 2 \sum\limits_{k\neq i} A_{iik} v_i u_k + \sum\limits_{\substack{j\neq i \\ k\neq i}} A_{ijk} u_j u_k = 0 \\ v_i(x,0) = \varphi_i(x), \quad i = 1,\dots,p, \quad x \in R, \quad t > 0 \end{cases}$$

then $B(\varphi,u) \geq 0$ if $\varphi \geq 0$, $u \geq 0$ and the solution is nonnegative]. Under condition (1.6) the entropy $\sum\limits_i \int u_i(x,t) \text{Log}\, u_i(x,t)dx$ is a nonincreasing function of t if the data have compact support.

Let $T > 0$; we want a bound of $u_i(x_0,T)$ depending only upon T and $\max\limits_i \|\varphi_i\|_{L^\infty(\mathbb{R})}$.

Let $J = [x_0 - t \max\limits_i C_i, x_0 - t \min\limits_i C_i]$ and define ψ_i by

$$(4.13)\qquad \psi_i(x) = \begin{cases} \varphi_i(x) & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}.$$

The solution v with initial data ψ will coincide with u at (x_0,T); on the other hand the solution has compact support for each time t with

$$(4.14)\qquad \begin{cases} \text{meas}\{x : v(x,t) \neq 0\} \leq kT \quad \text{for } 0 \leq t \leq T \\ v \text{ being the solution corresponding to } \psi. \end{cases}$$

Using the entropy we have $\sum\limits_i \int v_i(x,t) \text{Log}\, v_i(x,t)dx \leq \sum\limits_i \int \varphi_i(x) \text{Log}\, \varphi_i(x)dx$. Using $\lambda\, \text{Log}_+ \lambda - \frac{1}{e} \leq \lambda\, \text{Log}\, \lambda \leq \lambda\, \text{Log}_+ \lambda$ for $\lambda > 0$ we obtain

$$(4.15)\qquad \sum_i \int_R v_i(x,t) \text{Log}_+ v_i(x,t)dt \leq k(M_0 \text{Log}_+ M_0 + T)$$

(k designs various constants and $M_0 = \max\limits_i \|\varphi_i\|_{L^\infty}$). From (4.15) one deduces that there exists r depending only upon M_0 and T such that

$$(4.16)\qquad \sum_i \int_x^{x+r} |v_i(y,t)|dy \leq E_1, \quad \text{uniformly } x \in R,\ t \in [0,T].$$

Indeed if $w \geq 0$ and $\int w(y)\mathrm{Log}_+ w(y)dy \leq A$ we can decompose $\int_x^{x+r} w(y)dy$ into $I_1 + I_2$ where I_1 is the integral over the set of y such that $w(y) \geq R > 1$ and I_2 over the set of y such that $w(y) \leq R$; we bound I_1 by $\frac{A}{\mathrm{Log}_+ R}$ and I_2 by R_r; then one chooses R large enough so that $\frac{A}{\mathrm{Log}_+ R} \leq \frac{E_1}{2p}$ and then r small enough so that $rR \leq \frac{E_1}{2p}$. This shows that there exists r depending only upon A such that $\int_x^{x+r} w(y)dy \leq \frac{E_1}{p}$ uniformly in $x \in R$. Apply this to $u_i(\cdot,t)$ with $A = k(M_0 \mathrm{Log}_+ M_0 + T)$.

We use (4.16) and step 4 on domains D' constructed on interval basis $J' = \{y \in [x', x' + r]\}$ at time $t' \in [0,T]$, then on D' we have $\max_{i,(x,t)\in D'} |u_i(x,t)| \leq k_3 \max_{i,x\in J'} |u_i(x,t')|$. By moving $x' \in R$ we obtain

$$(4.17) \qquad M(t) \leq k_3 M(t') \quad \text{for } t' \leq t \leq t' + \frac{r}{\max\limits_i C_i - \min C_i}.$$

By applying (4.17) finitely many times one obtains a bound for $M(T)$ (which could be given more explicitly as a function of M_0 and T) and this proves Theorem 4.

4f: The first part of Theorem 5 follows from the fact that $u_i \in V_i$. Indeed $\frac{\partial v_i}{\partial t} = \frac{\partial u_i}{\partial t} + C_i \frac{\partial u_i}{\partial x} \in L^1(-\infty,+\infty,L^1(R))$ and $v_i(x,0) \in L^1(R)$ and so limits exists in $L^1(R)$ as t goes to $+\infty$ or $-\infty$. If $|u_j(x,t)| \leq M$ for each j, $x \in R$ and $t \geq 0$ we will bound $v_i(x,T + t) - v_i(x,T)$ by a quantity which tends to 0 as T goes to $+\infty$ uniformly for $t \geq 0$. For this we start from the fact that

$$u_i(y,T + t) - u_i(y - C_i t,T) = - \sum_{j,k} \int_0^t A_{ijk} u_j u_k(y - C_i s,T + t - s)ds$$

and we choose $y = x + C_i(T + t)$, so the left hand side is $v_i(x,T + t) - v_i(x,T)$. As in step 4 we bound $\int_0^t u_j u_k(y - C_i s,T + t - s)ds$ by $M|\int_0^t |u_j(y - C_i s,T + t - s)|ds|$ and we assume $C_i \neq C_j$. We know that $|u_j(z,\tau)| \leq w_j(z - C_j\tau)$ with $w_j \in L^1(R)$ and thus the integral is less than $M\int_0^t w_j(y - C_i s - C_j T - C_j t + C_j s)ds$ which with the choice of y is $M\int_T^{T+t} w_j(x + (C_i - C_j)\tau)d\tau$ which is less than $M\int_T^\infty w_j(x + (C_i - C_j)\tau))d\tau$ which goes to 0 as T goes to $+\infty$.

4.g: First we remark that, if the data φ have disjoint support, then $S_0(s)\varphi = S(s)\varphi$ for small s. More precisely let δ_{ij} be the distance between support (φ_i) and support (φ_j); then at time t the distance between support $(u_i(\cdot,t))$ and support $(u_j(\cdot,t))$ is more than $\max(0, \delta_{ij} - |t||C_i - C_j|)$. So if there exists $T > 0$ such that $\delta_{ij} \geq |C_i - C_j|T$ for all i,j then for $|t| \leq T$ we have $u_i u_j = 0$ if $C_i \neq C_j$ and, by using condition (S), u coincides with the solution of the linear equation.

Assume now that φ has compact support and let $\psi = S_0(-T)\varphi$ (we choose $T > 0$); then distance (support ψ_i, support ψ_j) $\geq |C_i - C_j|(T - \tau)$ for each i,j where τ depends upon the length of the support and the C_k's. For $T > \tau$ we have $S(T - \tau)\psi = S_0(T - \tau)\psi$ and so $S(T)S_0(-T)\varphi = S(\tau)S(T - \tau)\psi = S(\tau)S_0(T - \tau)\psi = S(\tau)S_0(-\tau)\varphi$ which is the desired limit as T goes to $+\infty$. Similarly when T goes to $-\infty$.

In the general case choose $\bar{\varphi}$ with compact support such that $\sum_j \|\varphi_j - \bar{\varphi}_j\|_{L^1} \leq \varepsilon$ (with $\sum_j \|\varphi_j\|_{L^1} \leq E_0$ and $\sum_j \|\bar{\varphi}_j\|_{L^1} \leq E_0$), then by using Theorem 1 ($S_0(t)$ being an isometry in L^1) we have $\|S(T)S_0(-T)\varphi - S(T)S_0(-T)\bar{\varphi}\|_{L^1} \leq k_2\varepsilon$ and so $S(T)S(-T)\varphi$ has strong limits as T goes to $+\infty$ or $-\infty$. This proves Theorem 6.

4h: $S(T)$ maps D_m into D_{m-1} for $m \geq 0$ by inequality (2.2) and has Lipschitz constant k_2 on D_1; as $S_0(s)$ is an isometry one obtains immediately the fact that $W_\pm$, $M_\pm$ map D_m into D_{m-1} for $m \geq 0$ and satisfy (2.7). Property (2.8) follows then easily: for example let $\varphi \in D_1$ and show that $M_+W_+\varphi = \varphi$. For $\varepsilon > 0$ one can find $s_0(\varepsilon)$ going to $+\infty$ as ε goes to 0 such that $\|M_+(W_+\varphi) - S(-s)S_0(s)W_+\varphi\| \leq \varepsilon$ for $s \geq s_0(\varepsilon)$ and $t_0(\varepsilon)$ going to $+\infty$ as ε goes to 0 such that $\|W_+\varphi - S_0(-t)S(t)\varphi\| \leq \varepsilon$ for $t \geq t_0(\varepsilon)$; this implies that

$$\|M_+W_+\varphi - S(-s)S_0(s)S_0(-t)S(t)\varphi\| \leq (1 + k_2)\varepsilon \quad \text{for } s \geq s_0(\varepsilon),\ t \geq t_0(\varepsilon)$$

one then chooses $s = t \geq \max(s_0(\varepsilon), t_0(\varepsilon))$ and let ε go to 0. The properties of S are then obvious: $SW_- = (W_+M_-)W_- = W_+(M_-W_-) = W_+$ on adequate domain, i.e. D_2, to have all operators defined.

5. Results without condition(s)

When condition (S) is not true we will use the sign condition

(1.3) $$A_{i,j,k} \leq 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad i \neq k$$

and replace condition (S) by one or two of the following conditions

(5.1) $$\begin{cases} \text{there exists nonnegative numbers } \lambda_i,\ i = 1,\dots,p \text{ such that} \\ b_{jk} = \sum_i \lambda_i A_{ijk} \text{ satisfies } b_{jk} \geq \sum_i |A_{ijk}| \text{ when } C_j = C_k . \end{cases}$$

(5.2) If $C_i = C_j = C_k$ then $A_{ijk} \geq 0$.

Remark that condition (S) implies (5.1) (with $\lambda_i = 0$ for each i) and (5.2). Some of the results of Theorems 1 to 7 then hold with some modifications.

Theorem 1': Assume (5.1) and (1.3) hold. Then there exists $E_0' > 0$, $k_1' > 0$ such that: if $\varphi_i \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with

(5.3) $$\begin{cases} \varphi_i \geq 0 \quad \text{for each } i \\ \sum_i \|\varphi_i\|_{L^1(\mathbb{R})} \leq E_0' \end{cases} .$$

Then on each interval $[0,T]$ where the solution exists one has

(5.4) $$u_i \geq 0 \quad \text{for each } i, \quad \text{a.e. } x \in R,\ t \in [0,T] ,$$

(5.5) $$\sum_i \left\| \frac{\partial u_i}{\partial t} + C_i \frac{\partial u_i}{\partial t} \right\|_{L^1(\mathbb{R}\times[0,1])} \leq k_1' \sum_i \|\varphi_i\|_{L^1(\mathbb{R})} . \qquad \blacksquare$$

Theorem 3': Assume (5.1), (5.2) and (1.3) hold. Then there exists $E_1' > 0$, $k_3' \geq 1$ such that: if $\varphi_i \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with

(5.6) $$\begin{cases} \varphi_i \geq 0 \quad \text{for each } i \\ \sum_i \|\varphi_i\|_{L^1(\mathbb{R})} \leq E_1' \end{cases}$$

then the solution exists for $t \in [0,\infty[$ and satisfies

(5.7) $$\sup_{t\geq 0, i} \|u_i(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq k_3' \max_i \|\varphi_i\|_{L^\infty} . \qquad \blacksquare$$

Theorem 4': Assume (5.1), (5.2), (1.3) and (1.6) hold. Then there exists a growth function $F(t,M)$ such that: if the data satisfy $0 \le \varphi_i(x) \le M$ a.e. for each i, then the solution exists for $t \in [0,\infty[$ and satisfies

$$0 \le u_i(x,t) \le F(t,M) \quad \text{a.e. for each } i . \qquad \blacksquare$$

Theorem 5': Assume (5.1) and (1.3) hold. Let $\varphi_i \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for each i satisfying

$$(5.8) \qquad \begin{cases} \varphi_i \ge 0 \quad \text{for each } i \\ \sum_i \|\varphi_i\|_{L^1(\mathbb{R})} \le E_0' \end{cases} .$$

Assume that the solution exists on $[0,+\infty[$. Let $v_i(x,t) = u_i(x + C_i t,t)$; then as t goes to $+\infty$ v_i has a strong limit in $L^1(\mathbb{R})$. If the solution stays bounded in $L^\infty(\mathbb{R})$ and if

$$(5.9) \qquad C_i = C_j = C_k \quad \text{implies} \quad A_{ijk} = 0$$

then v_i has a strong limit in $L^\infty(\mathbb{R})$. $\blacksquare$

Remark: Condition (1.3) is not enough for some of the results to hold. If one considers the Carleman equation

$$(5.10) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + u^2 - v^2 = 0, & u(x,0) = u_0(x) \\[2ex] \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} - u^2 + v^2 = 0, & v(x,0) = v_0(x) \end{cases}$$

then (1.3) holds and Theorem 3' holds with $E_1' = +\infty$ and $k_3' = 1$ but (5.5) is not true: if it was then $u(x + t,t)$ and $v(x - t,t)$ would have strong limits in $L^1(\mathbb{R})$ as t goes to $+\infty$ and, because $\int (u(x,t) + v(x,t))dx$ is constant these limits could not be zero if $\int (u_0(x) + v_0(x))dx > 0$; on the other hand Illner-Reed have shown a decay in $L^\infty(\mathbb{R})$ in $\frac{C}{t}$ with C depending only upon $\int (u_0(x) + v_0(x))dx$ (we give an alternate proof of this result in Appendix 2). $\blacksquare$

It would be interesting to know if some of the results fail for a good model in kinetic theory: i.e. one having conservation of mass, momentum and energy and satisfying the entropy condition (Carleman's model fails to conserve the momentum).

6. Proofs of Theorems 1' to 5'

6.a: We will suppose that the φ_i have compact support; the inequalities obtained will then easily be proved by approximation if $\varphi_i \in L^1(R) \cap L^\infty(R)$ for each i.

As proved in 4.e the solution satisfies $u_i(x,t) \geq 0$ a.e. $x \in R$, $t \in [0,T[$ for each i where T is the maximal time of existence in L^∞.

Let $S < T$, multiply the i-th equation in (1.1) by λ_i, sum in i and integrate in $x \in R$ and $t \in [0,S]$; we obtain

$$\sum_i \lambda_i \int_R u_i(x,S)\,dx + \sum_{i,j,k} \lambda_i A_{ijk} \int_R \int_0^S u_j u_k(x,t)\,dx\,dt = \sum_i \lambda_i \int_R \varphi_i(x)\,dx .$$

Using $u_i \geq 0$ and (5.1) we deduce

$$(6.1) \qquad \sum_{C_j = C_k} |A_{ijk}| \, \|u_j u_k\|_{L^1(R\times[0,S])} \leq M_1 \sum_i \|\varphi_i\|_{L^1(R)} + M_2 \sum_{C_j \neq C_k} \|u_j u_k\|_{L^1(R\times[0,S])} .$$

Let $\varepsilon_j(s) = \left\| \frac{\partial u_j}{\partial t} + C_j \frac{\partial u_j}{\partial x} \right\|_{L^1(R\times[0,S])}$ and remember that, by Lemma 1

$$\|u_j u_k\|_{L^1(R\times[0,S])} \leq \frac{1}{|C_j - C_k|} (\varepsilon_j(s) + \|\varphi_j\|_{L^1(R)})(\varepsilon_k(s) + \|\varphi_k\|_{L^1(R)})$$

for $C_j \neq C_k$.

We then deduce from (6.1)

$$(6.2) \qquad \sum_{j,k} \|u_j u_k\|_{L^1(R\times[0,S])} \leq M_3 \sum_i \|\varphi_i\|_{L^1(R)} + M_4\Big(\sum_i \|\varphi_i\|_{L^1(R)} + \sum_i \varepsilon_i(s)\Big)^2$$

and then by using (1.1)

$$(6.3) \qquad \sum_i \varepsilon_i(s) \leq M_4 \sum_i \|\varphi_i\|_{L^1(R)} + M_5\Big(\sum_i \|\varphi_i\|_{L^1(R)} + \sum_i \varepsilon_i(s)\Big)^2 .$$

When s goes to 0 $\varepsilon_i(s)$ goes to 0 and then, if we note $E = \sum_i \|\varphi_i\|_{L^1(R)}$, (6.3) implies that $\sum_i \varepsilon_i(s)$ is less than the smallest positive root of $X = M_4 E + M_5(E + X)^2$ and one checks easily that if $E \leq E_0' = \dfrac{M_4}{M_5(1 + 2M_4)^2}$ then the smallest root is less than $2M_4E$. This proves Theorem 1'.

6.b: As in 4.d we have to bound from above (we have 0 as lower bound)

$$\varphi_i(x - C_i t) - \sum_{j,k} \int_0^t A_{ijk} u_j u_k (x - C_i s, t - s) ds .$$

By (5.2) we can delete the terms for which $C_i = C_j = C_k$ because $A_{ijk} > 0$ and $u_j u_k > 0$; then there remains only terms which can be handled like in 4.d and this will prove Theorem 3'.

6.c: The proof of Theorem 4' is exactly similar to 4.e.

6.d: The proof of Theorem 5' is exactly similar to 4.f; by (5.9) the only integrals to bound can be handled like in 4.f.

Comments

The first results of global existence (without a bounded invariant region) was obtained by Mimura-Nishida [1]: using supplementary conservation laws they proved the analog of Theorem 3' for the Broadwell model. It was then recognized by Crandall-Tartar [2] how to use finite propagation speed and entropy to deduce Theorem 4' from Theorem 3'. The method was then applied to different classical models where Mimura-Nishida's argument could be carried on Cabannes [3], Leguillon [4].

REFERENCES

[1] Mimura-Nishida: Global solutions to the Broadwell's model of Boltzman equation for a simple discrete velocity gas, Proc. Intern. Symp. on Math. Prob. in Theor. Phys. RIMS Kyoto, January 1975.

[2] Tartar: Existence globale poir un système hyperbolique semilinéaire de la théorie cinétique des gaz, Séminaire Goulaouic-Schwarz 1975/76, no. 1, MR 57 #6865.

[3] Cabannes: Solution globale du problème de Cauchy en théorie cinétique discrète, J. Mécanique 17, 1978, no. 1, 1-22, MR 57 #18456.

[4] Leguillon: Solution globale d'un problème avec conditions aux limites en théorie cinétique discrète, Comptes. Rendus. Acad.Sci. Paris A-B 285, 1977, no. 16, 1125-1128, MR 58 #19720.

Appendix 1

Semilinear systems and continuity for the L^∞ weak-star topology.

Consider the following system in N dimensional space

$$
(1) \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + \operatorname{grad}_x u_i \cdot v_i + F_i(u_1,\dots,u_p), & x \in \mathbb{R}^N, \quad t \in I \\[2ex] u_i(x,0) = \varphi_i(x), & x \in \mathbb{R}^N, \quad i = 1,\dots,p \end{cases}
$$

where F_i are locally Lipschitz functions on $\mathbb{R}^p$ and v_i are vectors in $\mathbb{R}^N$. If the data $\varphi_i \in L^\infty(\mathbb{R}^N)$ with $\|\varphi_i\|_{L^\infty} \leq M$ for each i then there exists $t_0 > 0$ and a unique solution on $I = (-t_0,t_0)$ satisfying $\|u_i(\cdot,t)\|_{L^\infty} \leq 2M$ for $t \in [-t_0,t_0]$ and each i. The solution u depends continuously upon φ in the L^∞ strong topology.

We are interested here in the following question: for what functions F_i, $i = 1,\dots,p$ does the solution u depend continuously upon φ in the L^∞ weak-star topology restricted to the ball $\max_i \|\varphi_i\|_{L^\infty} \leq M$? (the weak-star topology restricted to a ball is metrizable). The answer to this question is given by the following:

Theorem: The solution depends continuously upon φ in the L^∞ weak-star topology (on bounded sets) if and only if

either a) $N > 1$ and each F_i is an affine function

or b) $N = 1$ and each F_i has the following form

$$
(2) \qquad F_i(u) = \sum_{j,k} A_{ijk} u_j u_k + \sum_j b_j u_j + c
$$

with

$$
(S) \qquad A_{ijk} = 0 \quad \text{if} \quad v_j = v_k .
$$

Proof: 1) It is enough to prove b) for $N = 1$.

Indeed if $N > 1$ taking φ_i of the form $\psi_i(x \cdot n)$ for some unit vector we are led to $N = 1$ with v_i replaced by $v_i \cdot n$. Then continuity will hold if F_i has form (2) with $A_{ijk} = 0$ when $v_j \cdot n = v_k \cdot n$; as for each couple j,k one can choose n such that this is true one sees that all coefficients A_{ijk} are 0 and so each F_i is affine. In that case sufficiency is obvious.

2) Take a sequence $\varphi_i^\varepsilon = \varphi_i(\frac{x}{\varepsilon})$ where each φ_i is periodic with period T_i so that as ε goes to 0 φ_i^ε converges in L^∞ weak-star to $\bar{\varphi}_i$ = average of φ_i. Then for $|t| \leq t_0$ we have uniformly in ε : $u_i^\varepsilon(x,t) = O(1)$ so $F_i(u^\varepsilon) = O(1)$ and, integrating (1) along a characteristic we obtain (replacing v_i by C_i like in (1.1))

(3) $$u_i^\varepsilon(x,t) = \varphi_i\left(\frac{x - C_i t}{\varepsilon}\right) + O(|t|)$$

which gives, as each F_i is locally Lipschitz

(4) $$F_i(u^\varepsilon(x,t)) = F_i\left(\varphi_1\left(\frac{x - C_1 t}{\varepsilon}\right),\dots,\varphi_p\left(\frac{x - C_p t}{\varepsilon}\right)\right) + O(|t|) .$$

If the weak-star continuity holds $u_i^\varepsilon(x,t)$ will converge to the solution starting from data $\bar{\varphi}_i$: let u_i^0 be this solution. If a subsequence of $F_i(u^\varepsilon)$ converges to w_i^0 then, by the equation, $\frac{\partial u_i^0}{\partial t} + \mathrm{grad}_x u_i^0 \cdot v_i + w_i^0 = 0$ and as u_i^0 is the solution we obtain $w_i^0 = F_i(u^0)$ and as $u_i^0 = \bar{\varphi}_i + O(|t|)$ we should have

(5) If $F_i(u^\varepsilon)$ converges in L^∞ weak-star to w_i^0 then $w_i^0 = F_i(\bar{\varphi}) + O|t|)$.

3) We will obtain the form for F_i by using different choices of φ_j. We now drop the index i of F_i.

Take $\varphi_2,\dots,\varphi_p$ to be constants, then

$$F\left(\varphi_i\frac{(x - C_1 t)}{\varepsilon},\varphi_2,\dots,\varphi_p\right) \text{ converges to average of } F(\varphi_1(y),\varphi_2,\dots,\varphi_p)$$

and by (5) this should be $F(\text{average } \varphi_1,\varphi_2,\dots,\varphi_p)$. So necessarily F is affine in φ_1 and similarly in φ_j for each j. So F is a combination of multilinear functions.

4) If $C_j = C_k$ there are no products $u_j u_k$ in F. Take $j = 1$, $k = 2$ and $\varphi_3,\dots,\varphi_p$ to be constants, then $F\left(\varphi_1\left(\frac{x - C_1 t}{\varepsilon}\right), \varphi_2\left(\frac{x - C_2 t}{\varepsilon}\right), \varphi_3,\dots,\varphi_p\right)$ converges to the average of $F(\varphi_1(y),\varphi_2(y),\varphi_3,\dots,\varphi_p)$ which should be $F(\text{average } \varphi_1, \text{ average } \varphi_2,\varphi_3,\dots,\varphi_p)$. If there was in F a term $\varphi_1\varphi_2 G(\varphi_3,\dots,\varphi_p)$ we would obtain a contradiction by taking $\varphi_1 = \varphi_2$ nonconstant except if $G(\varphi_3,\dots,\varphi_p) = 0$.

5) If $C_i \neq C_j \neq C_k \neq C_i$ there are no products $u_i u_j u_k$ in F. Take the indices to be 1, 2, 3. It will be enough to find periodic functions of period T_1, T_2, T_3, of average 0 such that $\varphi_1\left(\frac{x - C_1 t}{\varepsilon}\right)\varphi_2\left(\frac{x - C_2 t}{\varepsilon}\right)\varphi_3\left(\frac{x - C_3 t}{\varepsilon}\right)$ converges weakly to a nonzero constant.

Take $\varphi_1(y) = \sin \frac{y}{C_3 - C_1}$, $\varphi_2(y) = \cos \frac{y}{C_3 - C_2}$, $\varphi_3(y) = \sin \frac{(C_1 - C_2)y}{(C_3 - C_1)(C_3 - C_2)}$ then

as $\frac{x - C_1 t}{C_3 - C_1} - \frac{x - C_2 t}{C_3 - C_2} = \frac{(C_1 - C_2)(x - C_3 t)}{(C_3 - C_1)(C_3 - C_1)}$, the product is

$\varphi_1^2\left(\frac{x - C_1 t}{\varepsilon}\right)\varphi_2^2\left(\frac{x - C_2 t}{\varepsilon}\right) - \frac{1}{4} \sin 2 \frac{x - C_1 t}{\varepsilon(C_3 - C_1)} \sin 2 \frac{x - C_2 t}{\varepsilon(C_3 - C_2)}$ whose limit is $\frac{1}{4}$.

6) By 3), 4), and 5) each F_i must have form (2).

This condition is sufficient by a compensated-compactness argument:

$$(6)\quad \begin{cases} \text{If } v_\varepsilon(x,t),\ w_\varepsilon(x,t) \text{ converge in } L^\infty(w) \text{ weak-star to } v_0, w_0 \text{ with} \\[1em] \dfrac{\partial v_\varepsilon}{\partial t} + C \dfrac{\partial v_\varepsilon}{\partial x}, \text{ and } \dfrac{\partial w_\varepsilon}{\partial t} + C' \dfrac{\partial w_\varepsilon}{\partial x} \text{ bounded in } L^\infty(\Omega),\ \Omega \subset R^2 \text{ then} \\ \text{if } C \neq C' \ \ v_\varepsilon w_\varepsilon \text{ converges in } L^\infty(\Omega) \text{ weak-star to } v_0 w_0 \,. \end{cases}$$

For the proof and the motivation in constructing the test functions in 3), 4), and 5) see Tartar [1].

Tartar [1]. Compensated compactness and applications to partial differential equations, p. 136-212 in Nonlinear analysis and mechanics: Heriot Watt symposium vol. IV, R. J. Kreps ed. Research notes in Mathematics 39, Pitman.

Appendix 2

Decay for solution of Carleman's equation.

Theorem (Illner-Reed). Let u,v be solutions of Carleman's equation

$$(7)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + u^2 - v^2 = 0, & u(x,0) = u_0(x) \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} - u^2 + v^2 = 0, & v(x,0) = v_0(x) . \end{cases}$$

Assume $0 \leq u_0, v_0 \leq M$ a.e. and $\int_R (u_0(x) + v_0(x))dx = m < +\infty$. Then there exists a constant C_m depending upon m such that

$$(8)\qquad 0 \leq u, v \leq \min\left(M, \frac{C_m}{t}\right) \quad \text{for } t \in [0,\infty[.$$

Proof: It is a classical result that $0 \leq u, v \leq M$ for $t \in [0,\infty[$ (without assuming $m < +\infty$). Let m be given and define $G(t,M)$ as the supremum of $u(x,t)$ and $v(x,t)$ when u_0,v_0 satisfy $0 \leq u_0, v_0 \leq M$ and $\int (u_0 + v_0)dx = m$. Then $G(t,M) \leq M$ and $G(t,M)$ is nonincreasing in t. Trivially $G(s + t,M) \leq G(s,G(t,M))$ for every $s,t \geq 0$. If one now remarks that $\lambda u(\lambda x,\lambda t),\lambda v(\lambda x,\lambda t)$ satisfy Carleman's equation we see that $G(t,M) = \lambda G(\lambda t, \frac{M}{\lambda})$ for every $\lambda > 0$, so $G(t,M) = H(\frac{1}{t},M)$ with H positively homogeneous of order 1.

(8) is now a consequence of

Lemma 1: If $t \geq T > \frac{m}{2}$ then $G(t,1) \leq k$ with $k < 1$. ∎

Indeed $G(\frac{T}{M},M) = MG(t,1) \leq kM$. Then if $p \geq 1$ is an integer

$$G\left(\frac{T}{M}\left(1 + \frac{1}{k} + \cdots + \frac{1}{k^p}\right),M\right) \leq G\left(\frac{T}{M}\left(\frac{1}{M} + \frac{1}{k^2} + \cdots + \frac{1}{k^p}\right),kM\right)$$

$$= kG\left(\frac{T}{M}\left(1 + \frac{1}{k} + \cdots + \frac{1}{k^{p-1}}\right),M\right)$$

so by induction $G\left(\frac{T}{M}\left(1 + \frac{1}{k} + \cdots + \frac{1}{k^p}\right),M\right) \leq k^{p+1}M$ is $p \geq 0$ is an integer. If $t \geq \frac{T}{M}$ then for some $p \geq 0$ $\frac{T}{M}\left(1 + \frac{1}{k} + \cdots + \frac{1}{k^p}\right) \leq t \leq \frac{T}{M}\left(1 + \frac{1}{k} + \cdots + \frac{1}{k^{p+1}}\right)$ so $G(t) \leq k^{p+1}M \leq \frac{T}{t}(1 + k + \cdots + k^p) \leq \frac{T}{t(1 - k)}$. If $t \leq \frac{T}{M}$ then $G(t) \leq M \leq \frac{T}{t} \leq \frac{T}{t(1 - k)}$ so (8) holds with $C = \frac{T}{1 - k}$ and C only depends upon m. To prove Lemma 1 we will need 2 more lemmas:

Lemma 2: The solution of

$$\begin{cases} v' + v^2 = 1 \\ v(0) = \lambda \end{cases}$$

is given by $v(t,\lambda) = F(t,\lambda)$ which satisfies

$$0 \leq F(t,\lambda) \leq 1 + e^{-2t}(\lambda - 1) \quad \text{for} \quad \lambda \geq 0, \quad t \geq 0 \,. \tag{9}$$

Proof: The exact solution is $F(t,\lambda) = \dfrac{\lambda \text{ Cht} + \text{Sht}}{\lambda \text{ Sht} + \text{Cht}}$ which is concave in λ; $F(t,1) = 1$ and $\dfrac{\partial F}{\partial \lambda}(t,1) = e^{-2t}$. ■

Lemma 3: If

$$\begin{cases} u' + u^2 \leq 1 + e^{-2t}(\lambda(t) - 1) \\ 0 \leq u(0) \leq 1 \end{cases} \tag{10}$$

with

$$0 \leq \lambda(t) \leq 1 \quad \text{and} \quad \int_0^\infty \lambda(t)\,dt \leq m_0 \,. \tag{11}$$

Then $u(t) \leq \bar{u}(t)$ where $\bar{u}(t)$ satisfies:

$$\begin{cases} \bar{u}' + \bar{u}^2 = 1 \quad \text{on} \quad [0,m_0] \\ \qquad\qquad = 1 - e^{-2t} \quad \text{on} \quad [m_0 + \infty[\\ \bar{u}(0) = 1 \,. \end{cases} \tag{12}$$

Proof: $u(t) \leq w(t)$ where w satisfies

$$\begin{cases} w' + w^2 = 1 + e^{-2t}(\lambda(t) - 1) \\ w(0) = 1 \,. \end{cases}$$

Let $T < +\infty$ and maximize $w(T)$ under the constraints $0 \leq \lambda(t) \leq 1$ a.e. on $[0,T]$ and $\int_0^T \lambda(t)\,dt \leq m$. The set of constraints defines a compact set in $L^\infty(0,T)$ for the weak-star topology and the map $\lambda \to w(T)$ is continuous, so an optimal solution $\bar{\lambda}$ exists. We now identify $\bar{\lambda}$ from the necessary condition of optimality:

Let $\lambda = \bar{\lambda} + \delta\lambda$ satisfy the constraints; then to $\bar{\lambda} + \varepsilon\delta\lambda$ corresponds $\bar{w} + \varepsilon\delta w + 0(\varepsilon)$ where δw is given by

$$\begin{cases} (\delta w)' + 2\bar{w}\delta w = e^{-2t}\delta\lambda \\ \delta w(0) = 0 \end{cases}$$

so $\delta w(T) = \int_0^T a(t)\delta\lambda(t)\,dt$ where $a(t) = e^{-2t}\exp(-2\int_t^T \bar{w}(s)\,ds)$. As $\bar{w}(T)$ is maximum $\delta w \leq 0$ for all admissible choices of λ and this says that $\int_0^T a(t)\lambda(t)\,dt$ is maximum for $\bar{\lambda}$. This characterizes $\bar{\lambda}$ because a is nonincreasing: $a'(t) = e^{-2t}\exp(-2\int_t^T \bar{w}(s)\,ds)[2\bar{w}(t) - 2] \leq 0$ as $0 \leq \bar{w} \leq 1$. The optimal solution is $\bar{\lambda} = 1$ on $[0,m_0]$, 0 on $[m_0,T]$ (with modification $\bar{\lambda} \equiv 1$ if $T \leq m_0$). So the optimal value is $\bar{u}(T)$ (note that $\bar{u} = 1$ on $[0,m_0]$). ∎

<u>Proof of Lemma 1</u>: Let $0 \leq u_0, v_0 \leq 1$. So $0 \leq u, v \leq 1$ for $t \geq 0$. As $\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} + v^2 \leq 1$ one uses Lemma 2 to get

$$v(x,t) \leq F(v_0(x+t)) \leq 1 + e^{-2t}(v_0(x+t) - 1) .$$

Now look at u on the characteristic $x = x_0 + t$:

$$\frac{du}{dt} + u^2 = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u^2 = v^2 \leq F^2(v_0(x_0+t)) \leq F \leq 1 + e^{-2t}(v_0(x_0+t) - 1)$$

which corresponds to $\lambda(t) = v_0(x+2t)$ and so $\int_0^\infty \lambda(t)\,dt = \frac{1}{2}\int v_0(y)\,dy \leq \frac{m}{2}$. By Lemma 2 we conclude that $u(x_0+t,t) \leq \bar{u}(t)$ where $\bar{u}$ is constructed by (12) with $m_0 = \frac{m}{2}$. Exchanging the roles of u and v we get $G(t,1) \leq \bar{u}(t)$ and clearly $\bar{u} < 1$ for $t > m_0 = \frac{m}{2}$. ∎

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2164	2. GOVT ACCESSION NO. AD-A096648
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) 6 SOME EXISTENCE THEOREMS FOR SEMILINEAR HYPERBOLIC SYSTEMS IN ONE SPACE VARIABLE.	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Luc C. Tartar (14) MRC-TSR-2164	8. CONTRACT OR GRANT NUMBER(s) (15) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number #1 (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709 (11)	12. REPORT DATE Jan[redacted]81
	13. NUMBER OF PAGES 27 (12) 31
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 9) Technical summary rept.	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Semilinear hyperbolic system
Scattering
Kinetic theory
Global existence
Asymptotic behaviour

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider semilinear hyperbolic systems in one space variable of the type

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} + \sum\limits_{j,k} A_{ijk} u_j u_k = 0, & x \in \mathbb{R},\ t \in \mathbb{R},\ i = 1,\dots,p \\ u_i(x,0) = \varphi_i(x), & x \in \mathbb{R},\ i = 1,\dots,p\ . \end{cases}$$

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 221200

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT - Cont'd.

We first introduce a special condition

(S) $$A_{ijk} = 0 \quad \text{if} \quad c_j = c_k \; .$$

Under condition (S) we prove: local existence and uniqueness if the data are in $L^1(\mathbb{R})$; global existence, L^∞ stability and the existence of wave operators and of a scattering operator when the data have small norm in $L^1(\mathbb{R})$.

Adding a sign condition

(s) $$A_{ijk} \leq 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad i \neq k$$

and the entropy condition

(E) $\sum_{i,j,k} A_{ijk}\lambda_j\lambda_k \operatorname{Log} \lambda_i \geq 0$ for all $\lambda \in \mathbb{R}^p$ such that $\lambda_i > 0$ for each i we obtain global existence if the data are nonnegative and in $L^\infty(\mathbb{R})$.

We then replace condition (S) by a weaker one and obtain some of the above results in that case.